Most Maximally Monotone Operators Have a Unique Zero and a Super-regular Resolvent

Xianfu Wang*

January 21, 2013

Abstract

Maximally monotone operators play important roles in optimization, variational analysis and differential equations. Finding zeros of maximally monotone operators has been a central topic. In a Hilbert space, we show that most resolvents are super-regular, that most maximally monotone operators have a unique zero and that the set of strongly monotone mapping is of first category although each strongly monotone operator has a unique zero. The results are established by applying the Baire Category Theorem to the space of nonexpansive mappings.

2010 Mathematics Subject Classification: Primary 47H05, 47H10; Secondary 54E52, 47H09, 54E50.

Keywords: Asymptotic regularity, Baire Category, fixed point, graphical convergence, maximally monotone operator, nonexpansive mapping, resolvent, reflected resolvent, super-regularity, zeros of monotone operator, weakly contractive mapping.

1 Introduction

Throughout, X is a real Hilbert space whose inner product is denoted by $\langle x, y\rangle$ and induced inner product norm by $\|x\| := \sqrt{\langle x, x\rangle}$ for $x, y \in X$. Recall that a set-valued operator $A\colon X \rightrightarrows X$ (i.e., $(\forall x \in X)\ Ax \subseteq X$) with graph $\operatorname{gr} A$ is *monotone* if

$$(1)\qquad (\forall (x,u) \in \operatorname{gr} A)(\forall (y,v) \in \operatorname{gr} A) \quad \langle x-y, u-v\rangle \geq 0$$

*Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: shawn.wang@ubc.ca.

where $\operatorname{gr} A := \{(x,y) \in X \times X : \; y \in Ax\}$, and that A is *maximally monotone* if it is impossible to find a proper extension of A that is still monotone. We call $A : X \rightrightarrows X$ *strongly monotone* [5, 27] if there exists $\varepsilon > 0$ such that $A - \varepsilon \operatorname{Id}$ is monotone in which $\operatorname{Id}\colon X \to X\colon x \mapsto x$ denotes the *identity operator*.

We shall work in the *space of nonexpansive mappings* defined on X, i.e.,

$$\mathcal{N}(X) := \{T : X \to X : \; \|Tx - Ty\| \leq \|x - y\|, \forall\, x, y \in X\};$$

the *space of firmly nonexpansive mappings*

$$\mathcal{J}(X) := \{T : X \to X : \; \|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \forall\, x, y \in X\};$$

and the space of *maximal monotone operators*

$$\mathcal{M}(X) := \{A : X \rightrightarrows X : \; A \text{ is maximally monotone}\}$$

endowed with a metric defined in Section 2. The reason to investigate nonexpansive mappings defined on X is that they are directly related to maximally monotone operators.

In this note, we study generic properties of $\mathcal{N}(X)$, $\mathcal{M}(X)$ and $\mathcal{J}(X)$ by the Baire Category Theorem. A recent result due to Reich and Zaslavski implies that most nonexpansive mappings in $\mathcal{N}(X)$ are super-regular so that each of them has a unique fixed point. Utilizing Reich and Zaslavaski's technique, we show that (i) Most resolvents in $\mathcal{J}(X)$ are super regular, thus asymptotically regular; (ii) Most maximally monotone operators in $\mathcal{M}(X)$ have a unique zero; (iii) The set of strongly monotone operators is only a first category set in $\mathcal{M}(X)$ even though it is dense.

While extensive study has been done on $\mathcal{N}(X)$ [5, 14, 15, 7, 9, 19, 21, 23, 24] and on $\mathcal{M}(X)$ [1, 5, 27, 29, 31], generic properties on $\mathcal{M}(X)$ and $\mathcal{J}(X)$ seem new. They are particularly interesting for the optimization field. Note that De Blasi and Myjak only considered generic properties of continuous bounded monotone operators on a bounded set in [8].

In the reminder of this section, we introduce some definitions, basic facts and preliminary results. For $A \in \mathcal{M}(X)$, we define its *resolvent* and *reflected resolvent* (or Cayley transform) by

$$J_A := (A + \operatorname{Id})^{-1}, \quad R_A := 2J_A - \operatorname{Id}.$$

It is well-known that $J_A + J_{A^{-1}} = \operatorname{Id}$, $R_A + R_{A^{-1}} = 0$, see, e.g., [27], [4, Proposition 4.1]. Both resolvent and reflected resolvent play a key role in the proximal point algorithm and Douglus-Rachford algorithm [5, 25, 11, 12, 17, 4].

The following well-known characterizations about firmly nonexpansive mappings, nonexpansive mappings and maximally monotone operators are crucial.

Fact 1.1 (See, e.g., [5, 15, 14].) *Let $T\colon X \to X$. Then the following are equivalent:*

(i) *T is firmly nonexpansive.*

(ii) $2T - \mathrm{Id}$ *is nonexpansive.*

(iii) $(\forall x \in X)(\forall y \in X)\ \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$.

(iv) $(\forall x \in X)(\forall y \in X)\ 0 \leq \langle Tx - Ty, (\mathrm{Id} - T)x - (\mathrm{Id} - T)y\rangle$.

Fact 1.2 (Eckstein & Bertsekas, Minty) [18, 31, 13] *Let $A : X \rightrightarrows X$ be monotone. Then A is maximally monotone if and only if J_A is firmly non-expansive and has a full domain.*

For $T : X \to X$, let $\operatorname{Fix} T$ denote its fixed point set $\operatorname{Fix} T := \{x \in X : \ Tx = x\}$. Facts 1.1, 1.2 allow us to summarize the relationship among $\mathcal{N}(X), \mathcal{J}(X), \mathcal{M}(X)$.

Proposition 1.3 (i) $\mathcal{N}(X) = \{R_A : A \in \mathcal{M}(X)\}$,

$$\mathcal{M}(X) = \left\{ \left(\frac{T + \mathrm{Id}}{2}\right)^{-1} - \mathrm{Id} : \ T \in \mathcal{N}(X) \right\}.$$

(ii) $\mathcal{J}(X) = \{J_A : \ A \in \mathcal{M}(X)\}$,

$$\mathcal{M}(X) = \left\{ T^{-1} - \mathrm{Id} : \ T \in \mathcal{J}(X) \right\}.$$

(iii) $\mathcal{N}(X) = \{2T - \mathrm{Id} : \ T \in \mathcal{J}(X)\}$,

$$\mathcal{J}(X) = \left\{ \frac{T + \mathrm{Id}}{2} : \ T \in \mathcal{N}(X) \right\}.$$

(iv) *Let $A \in \mathcal{M}(X)$. Then* $\operatorname{Fix} R_A = \operatorname{Fix} J_A = A^{-1}(0)$.

Many nice properties and applications about $\mathcal{N}(X), \mathcal{J}(X), \mathcal{M}(X)$ can be found in [1, 5, 6, 23, 24] and they continue to flourish. We refer readers to [6] for a systematic relationship among these three spaces. Let us now turn to the graphical convergence of set-valued maximal monotone operstors.

Definition 1.4 [1, page 360] *Given a sequence of maximally monotone operators*

$$\{A_n : \ n \in \mathbb{N}\}, A.$$

The sequence $\{A_n : \ n \in \mathbb{N}\}$ is said to be graphically convergent to A, written as $A_n \xrightarrow{g} A$, if

> *for every $(x, y) \in \operatorname{gr} A$ there exists $(x_n, y_n) \in \operatorname{gr} A_n$ such that $x_n \to x, y_n \to y$ strongly in $X \times X$.*

In terms of set convergence $\operatorname{gr} A \subset \liminf \operatorname{gr} A_n$.

Proposition 1.5 *The following are equivalent*

(i) *A sequence of maximally monotone operators $(A_k)_{k=1}^{\infty}$ in $\mathcal{M}(X)$ converges graphically to A;*

(ii) *$(J_{A_k})_{k=1}^{\infty}$ converges pointwise to J_A on X;*

(iii) *$(R_{A_k})_{k=1}^{\infty}$ converges pointwise to R_A on X.*

Proof. (i)$\Leftrightarrow$ (ii) follows from [1, Proposition 3.60, pages 361-362]. (ii)$\Leftrightarrow$ (iii) is obvious since $R_A = 2J_A - \mathrm{Id}$. ■

A set S in a complete metric space Y is called *residual* if there is a sequence of dense and open sets $O_n \subset Y$ such that $\bigcap_{n=1}^{\infty} O_n \subset S$; in this case we call $\bigcap_{n=1}^{\infty} O_n$ a dense G_δ set. A classical theorem of Baire is

Fact 1.6 (Baire Category Theorem) [26, page 158] *Let Y be a complete metric space and $\{O_n\}$ a countable collection of dense open subsets of Y. Then $\bigcap_{n=1}^{\infty} O_n$ is dense in Y.*

The technique of Baire Category has been instrumental in studying fixed point of nonexpansive mappings; see, e.g., [7, 8, 9, 19, 20, 21, 23, 24].

The paper is organized as follows. In Section 2 we give the main result. In Section 3 we introduce a class of weakly contractive mappings which contains contractive mappings, and show that although it is dense, it is only a set of first category.

Notation. For a set-valued mapping $A : X \rightrightarrows X$, we write $\operatorname{dom} A := \{x \in X \mid Ax \neq \varnothing\}$ and $\operatorname{ran} A := A(X) = \bigcup_{x \in X} Ax$ for the *domain* and *range* of A, respectively. $\mathbf{B}_r(x)$ denotes the closed ball of radius r centered at x. $\mathbb{N}$ stands for the set of natural numbers.

2 Main results

In this section, using Reich and Zaslavski's technique on super-regular mappings we establish a generic property of super-regular mappings in complete subspaces of $(\mathcal{N}(X), \rho)$. This allows us to show that most resolvents are super-regular; most maximally monotone operators have a super-regular reflected resolvent and a unique zero.

We start with three complete metric spaces which set up the stage for the Baire Category Theorem.

On $\mathcal{N}(X)$ we define a metric, for $T_1, T_2 \in \mathcal{N}(X)$

$$\rho(T_1, T_2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|T_1 - T_2\|_n}{1 + \|T_1 - T_2\|_n} \tag{2}$$

where $\|T_1 - T_2\|_n := \sup_{\|x\| \le n} \|T_1 x - T_2 x\|$. The metric ρ defines a topology of pointwise convergence on X and uniform convergence on bounded subsets of X.

Proposition 2.1 *$(\mathcal{N}(X), \rho)$ is a complete metric space.*

Proof. It is easy to see that ρ is a metric (cf. [16, pages 10-11]). We show that $\mathcal{N}(X)$ is complete. Assume that $(T_k)_{k=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{N}(X), \rho)$. Then for every $n \in \mathbb{N}$, $(T_k)_{k=1}^{\infty}$ is a uniform Cauchy sequence on $\mathbf{B}_n(0)$. In particular, $(T_k(x))_{k=1}^{\infty}$ is Cauchy in X for each $x \in \mathbf{B}_n(0)$, so $T_k(x)$ converges to $Tx \in X$. Moreover, for every $n \in \mathbb{N}$, $\|T_k - T\|_n \to 0$ as $k \to \infty$. Since each T_k is nonexpansive, T is nonexpansive, i.e., $T \in \mathcal{N}(X)$. It remains to show $\rho(T_k, T) \to 0$ as $k \to \infty$. Let $\varepsilon > 0$. Choose $M \in \mathbb{N}$ large such that

$$\sum_{n=M+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}.$$

For this M, choose a large $N \in \mathbb{N}$ such that $\|T_k - T\|_M < \frac{\varepsilon}{2}$ when $k > N$. Then for $k > N$ we have

$$\rho(T_k, T) = \sum_{n=1}^{M} \frac{1}{2^n} \frac{\|T_k - T\|_n}{1 + \|T_k - T\|_n} + \sum_{n=M+1}^{\infty} \frac{1}{2^n} \frac{\|T_k - T\|_n}{1 + \|T_k - T\|_n} \tag{3}$$

$$\le \sum_{n=1}^{M} \frac{1}{2^n} \frac{\varepsilon/2}{1 + \varepsilon/2} + \sum_{n=M+1}^{\infty} \frac{1}{2^n} < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{4}$$

Hence $(\mathcal{N}(X), \rho)$ is complete. ■

Remark 2.2 In [21], Reich and Zaslavaski define a unform space $(\mathcal{N}(X), \mathcal{U})$ where the uniformity $\mathcal{U}$ is defined by the base

$$E(n, \varepsilon) = \{(T, S) \in \mathcal{N}(X) \times \mathcal{N}(X) : \ \|T - S\|_n < \varepsilon\}$$

for $n \in \mathbb{N}, \varepsilon > 0$. The topology induced by this uniformity and the metric ρ are exactly the same.

On $\mathcal{M}(X)$ let us define a metric

$$\tilde{\rho}(A, B) := \rho(R_A, R_B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|R_A - R_B\|_n}{1 + \|R_A - R_B\|_n} \tag{5}$$

for $A, B \in \mathcal{M}(X)$.

Proposition 2.3 (i) *The space of monotone operators $(\mathcal{M}(X), \tilde{\rho})$ is a complete metric space, and it is isometric to $(\mathcal{N}(X), \rho)$.*

(ii) *When $X = \mathbb{R}^N$, the topology on $(\mathcal{M}(X), \tilde{\rho})$ is precisely the topology of graphical convergence.*

Proof. (i) By Facts 1.1, 1.2, under the mapping $A \mapsto R_A$

$$(\mathcal{M}(X), \tilde{\rho}) \text{ and } (\mathcal{N}(X), \rho) \text{ are isometric.}$$

Since $(\mathcal{N}(X), \rho)$ is complete by Proposition 2.1, we conclude that $(\mathcal{M}(X), \tilde{\rho})$ is complete.

(ii) When $X = \mathbb{R}^N$, on $\mathcal{N}(X)$ pointwise convergence and uniform convergence on compact subsets are the same. By Proposition 1.5, we obtain that the topology on $(\mathcal{M}(X), \tilde{\rho})$ is exactly the topology of graphical convergence. ■

On $\mathcal{J}(X)$ let us define a metric

$$\hat{\rho}(T_1, T_2) := \rho(2T_1 - \mathrm{Id}, 2T_2 - \mathrm{Id}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|2T_1 - 2T_2\|_n}{1 + \|2T_1 - 2T_2\|_n} \tag{6}$$

for $T_1, T_2 \in \mathcal{J}(X)$.

Proposition 2.4 *The space of resolvents* $(\mathcal{J}(X), \hat{\rho})$ *is a complete metric space, and it is isometric to* $(\mathcal{N}(X), \rho)$.

Proof. By Fact 1.1, under the mapping $T \mapsto 2T - \mathrm{Id}$

$$(\mathcal{J}(X), \hat{\rho}) \text{ and } (\mathcal{N}(X), \rho) \text{ are isometric.} \tag{7}$$

Since $(\mathcal{N}(X), \rho)$ is complete by Proposition 2.1, the result holds. ■

Next we study the denseness of contraction mappings and strongly monotone operators, which are required in later proofs.

Definition 2.5 *The map* $T \in \mathcal{N}(X)$ *is called a contraction with modulus* $1 > l \geq 0$ *if*

$$\|Tx - Ty\| \leq l\|x - y\| \; \forall \, x, y \in X.$$

Lemma 2.6 (i) *(***denseness of contraction mappings***) In* $(\mathcal{N}(X), \rho)$ *the set of contractions is dense, i.e., for very* $\varepsilon > 0$ *and* $T \in \mathcal{N}(X)$ *there exists a contraction* $T_1 \in \mathcal{N}(X)$ *such that* $\rho(T, T_1) < \varepsilon$.

(ii) *(***denseness of contractive firmly nonexpansive mappings***) In* $(\mathcal{J}(X), \hat{\rho})$ *the set of contraction is dense, i.e., for very* $\varepsilon > 0$ *and* $T \in \mathcal{J}(X)$ *there exists a contraction* $T_1 \in \mathcal{J}(X)$ *such that* $\hat{\rho}(T, T_1) < \varepsilon$.

Proof. (i) Let $T \in \mathcal{N}(X)$ and $1 > \varepsilon > 0$. Choose an integer M sufficiently large such that

$$\sum_{n=M+1}^{\infty} \frac{1}{2^n} \leq \frac{\varepsilon}{2}. \tag{8}$$

Choose

$$0 < \lambda < \frac{\varepsilon}{2(1 + \|T\|_M)} < \frac{1}{2}$$

and define

$$T_1 := (1 - \lambda)T.$$

Then T_1 is a contraction with modulus $1/2 < 1 - \lambda < 1$. As

$$\|T_1 - T\|_M = \sup_{\|x\| \leq M} \|(1 - \lambda)Tx - Tx\| \tag{9}$$

$$= \lambda \sup_{\|x\| \leq M} \|Tx\| = \lambda \|T\|_M < \frac{\varepsilon}{2}. \tag{10}$$

Using $\|T_1 - T\|_n \leq \|T_1 - T\|_M < \frac{\varepsilon}{2}$ when $n \leq M$ and (8), we have

$$\rho(T_1, T) = \sum_{n=1}^{M} \frac{1}{2^n} \frac{\|T_1 - T\|_n}{1 + \|T_1 - T\|_n} + \sum_{n=M+1}^{\infty} \frac{1}{2^n} \frac{\|T_1 - T\|_n}{1 + \|T_1 - T\|_n} \tag{11}$$

$$\leq \sum_{n=1}^{M} \frac{1}{2^n} \frac{\varepsilon}{2} + \sum_{n=M+1}^{\infty} \frac{1}{2^n} \tag{12}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{13}$$

so $\rho(T, T_1) < \varepsilon$.

(ii) The proof is similar as in (i) by replacing ρ by $\hat{\rho}$ and by observing that $T_1 = (1 - \lambda)T \in \mathcal{J}(X)$ if $T \in \mathcal{J}(X)$ and $0 \leq \lambda \leq 1$. ■

To study monotone operators, we need:

Fact 2.7 [6, Corollary 4.7] *Let $A : X \rightrightarrows X$ be maximally monotone. Then the following are equivalent:*

(i) *Both A and A^{-1} are strongly monotone;*

(ii) *There exists $\varepsilon > 0$ such that both $(1 + \varepsilon)J_A$ and $(1 + \varepsilon)J_{A^{-1}}$ are firmly nonexpansive;*

(iii) *R_A is a Banach contraction.*

Lemma 2.8 (denseness of strongly monotone mappings) *In $(\mathcal{M}(X), \tilde{\rho})$ the set of monotone operators A such that both A and A^{-1} are strongly monotone is dense, i.e., for very $\varepsilon > 0$ and $A \in \mathcal{M}(X)$ there exists a $B \in \mathcal{M}(X)$ such that both B and B^{-1} are strongly monotone, and $\tilde{\rho}(A, B) < \varepsilon$. Consequently, the set of strongly monotone operators is dense in $\mathcal{M}(X)$.*

Proof. Under the mapping $A \mapsto R_A$

$$(\mathcal{M}(X), \tilde{\rho}) \text{ and } (\mathcal{N}(X), \rho) \text{ are isometric.}$$

Let $A \in \mathcal{M}(X)$ and $\varepsilon > 0$. By Lemma 2.6(i), for R_A there exists a contraction T_1 such that $\rho(R_A, T_1) < \varepsilon$. Proposition 1.3(i) says that there exists $B \in \mathcal{M}(X)$ such that $T_1 = R_B$. By Fact 2.7 both B, B^{-1} are strongly monotone. The proof is complete by using $\tilde{\rho}(A, B) = \rho(R_A, R_B)$. ■

To prove our main results, we require *super-regular mappings* introduced by Reich and Zaslavaski [21].

Definition 2.9 (Reich-Zaslavski) *A mapping $T : X \to X$ is called super-regular if there exists a unique $x_T \in X$ such that for each $s > 0$, when $n \to \infty$,*

$$T^n x \to x_T \quad \textit{uniformly on } \mathbf{B}_s(0).$$

Our next two results collect some elementary properties of super-regular mappings.

Proposition 2.10 *Assume that $T : X \to X$ is super-regular and continuous. Then* $\operatorname{Fix} T$ *is a singleton.*

Proof. Let $x \in X$. Using the continuity and super-regularity of T, we have

$$x_T = \lim_{n \to \infty} T^n x = \lim_{n \to \infty} T(T^{n-1}x) = T x_T$$

so $x_T \in \operatorname{Fix} T$. Let $x \in \operatorname{Fix} T$. By the super-regularity of T and $T^n x = x$, $x = \lim_{n \to \infty} T^n x = x_T$. Hence $\operatorname{Fix} T = \{x_T\}$. ■

Proposition 2.11 (i) *If $T \in \mathcal{N}(X)$ is a contraction, then T is super-regular.*

(ii) *If $A \in \mathcal{M}(X)$ has both A and A^{-1} being strongly monotone, then R_A and J_A are super-regular.*

Proof. (i) Let $s > 0$. Let T be a contraction with modulus $0 \leq l < 1$. By the Banach Contraction Principle [16, pages 300-302], T has a unique fixed point x_T, and with arbitrary $x \in X$ the error estimate is

$$\|T^n x - x_T\| \leq \frac{l^n}{1-l} \|x - Tx\|.$$

For every $x \in \mathbf{B}_s(0)$,

$$\|T^n x - x_T\| \leq \frac{l^n}{1-l} (\|x\| + \|Tx - T0\| + \|T0\|) \leq \frac{l^n}{1-l} (s + ls + \|T0\|).$$

Therefore,

$$\|T^n - x_T\|_s \leq \frac{l^n}{1-l}(s + ls + \|T0\|) \to 0 \quad \text{when } n \to \infty.$$

Since $s > 0$ was arbitrary, T is super-regular.

(ii) By Fact 2.7, R_A is a contraction. Since A is strongly monotone, J_A is a contraction [25]. Hence (i) applies. ■

The proof ideas to Proposition 2.12 and Theorem 2.13 below are due to Reich and Zaslaski [21, 20]. We adopt them to our complete metric space setting, and to subspaces of $\mathcal{N}(X)$. For two metrics ρ, d on $\mathcal{F} \subset \mathcal{N}(X)$, if $\rho(T_1, T) \leq d(T_1, T)$ for all $T_1, T \in \mathcal{F}$ we write $\rho \leq d$.

Proposition 2.12 *Assume that $\mathcal{F} \subseteq \mathcal{N}(X)$, $(\mathcal{F}, d)$ is complete and $d \geq \rho$. Let $T \in \mathcal{F}$ be super-regular and ε, s be positive numbers. Then there exists $\delta > 0$ and $n_0 \geq 2$ such that when $d(T_1, T) < \delta$ and $n \geq n_0$ we have*

$$\|T_1^n x - x_T\| < \varepsilon \quad \textit{for every } x \in \mathbf{B}_s(0), \tag{14}$$

i.e., $\|T_1^n - x_T\|_s < \varepsilon$.

Proof. We may and do assume that $0 < \varepsilon < 1/2$. Let x_T denote the unique fixed point of T. Choose an integer $M > 1 + 2s + 4\|x_T\|$ so that

$$s < \frac{M}{2}, \qquad \frac{1}{2} + s + 2\|x_T\| < \frac{M}{2}. \tag{15}$$

As T is super-regular, there exists $n_0 \geq 2$ such that

$$\|T^n x - x_T\| < \frac{\varepsilon}{8} \quad \text{whenever } x \in \mathbf{B}_M(0) \text{ and } n \geq n_0. \tag{16}$$

Put

$$\delta := \frac{1}{2^M}\left(\frac{(8n_0)^{-1}\varepsilon}{1 + (8n_0)^{-1}\varepsilon}\right).$$

We will show that (14) holds when $d(T_1, T) < \delta$ and $n \geq n_0$.

Let $d(T_1, T) < \delta$. Then $\rho(T_1, T) < \delta$. Using the definition of ρ and that $t \mapsto \frac{t}{1+t}$ is strictly increasing on $[0, +\infty)$, we have

$$\|T_1 - T\|_M < (8n_0)^{-1}\varepsilon. \tag{17}$$

Claim 1. Whenever $x \in \mathbf{B}_{M/2}(0)$ and $1 \leq n \leq n_0$,

$$\|T_1^n x - T^n x\| < n(8n_0)^{-1}\varepsilon, \tag{18}$$

$$\|T_1^n x\| < \frac{1}{2} + \|x\| + 2\|x_T\| < M. \tag{19}$$

We prove this by induction. As $T \in \mathcal{N}(X)$ and $Tx_T = x_T$, for every $n \in \mathbb{N}$,

$$\|T_1^n x - T^n x\| \le \|T_1^n x - TT_1^{n-1}x\| + \|TT_1^{n-1}x - T^n x\| \tag{20}$$
$$\le \|T_1^n x - TT_1^{n-1}x\| + \|T_1^{n-1}x - T^{n-1}x\|, \tag{21}$$

and

$$\|T_1^n x - x_T\| \le \|T_1^n x - T^n x\| + \|T^n x - x_T\| \tag{22}$$
$$\le \|T_1^n x - T^n x\| + \|x - x_T\| \tag{23}$$
$$\le \|T_1^n x - T^n x\| + \|x\| + \|x_T\|. \tag{24}$$

Now when $n = 1$, (18) follows from (17); for (19), by (22) and (17)

$$\|T_1 x\| \le \|T_1 x - x_T\| + \|x_T\| \le \|T_1 x - Tx\| + \|x\| + 2\|x_T\| < \frac{1}{2} + \|x\| + 2\|x_T\|.$$

Assume that (18)-(19) hold for $1 \le n < n_0$, i.e.,

$$\|T_1^n x - T^n x\| < n(8n_0)^{-1}\varepsilon, \tag{25}$$
$$\|T_1^n x\| < \frac{1}{2} + \|x\| + 2\|x_T\| < M. \tag{26}$$

Using (20) for $n+1$, (25), (17), $\|T_1^n x\| < M$ and $n < n_0$, we obtain

$$\|T_1^{n+1}x - T^{n+1}x\| \le \|T_1^{n+1}x - TT_1^n x\| + \|T_1^n x - T^n x\| \tag{27}$$
$$< (8n_0)^{-1}\varepsilon + n(8n_0)^{-1}\varepsilon = (n+1)(8n_0)^{-1}\varepsilon. \tag{28}$$

Using (22) for $n+1$, (27),

$$\|T_1^{n+1}x\| \le \|T_1^{n+1}x - x_T\| + \|x_T\| \tag{29}$$
$$\le \|T_1^{n+1}x - T^{n+1}x\| + \|x\| + 2\|x_T\| \tag{30}$$
$$< (n+1)(8n_0)^{-1}\varepsilon + \|x\| + 2\|x_T\| < \frac{1}{2} + \|x\| + 2\|x_T\|. \tag{31}$$

This establishes (18)-(19).

Claim 2.

$$\|T_1^n y - x_T\| < \varepsilon \qquad \text{whenever } y \in \mathbf{B}_s(0) \text{ and } n \ge n_0. \tag{32}$$

This is done again by induction. When $n = n_0$, as $\|y\| \le s < M/2$, by (16) and (18)

$$\|T_1^{n_0}y - x_T\| \le \|T_1^{n_0}y - T^{n_0}y\| + \|T^{n_0}y - x_T\| < \varepsilon/8 + \varepsilon/8 < \varepsilon.$$

Assume that (32) holds for all $n_0 \le n \le k$. For $i = 1, \dots, n_0$, (19) and (15) give

$$\|T_1^i y\| < 1/2 + \|y\| + 2\|x_T\| < 1/2 + s + 2\|x_T\| < M/2; \tag{33}$$

For $k \geq i > n_0$, (32) and (15) give

$$\|T_1^i y\| \leq \|T_1^i y - x_T\| + \|x_T\| < 1/2 + \|x_T\| < M/2. \tag{34}$$

Set $j = k + 1 - n_0$ and $x = T_1^j y$. Then $1 \leq j < k$ and $\|x\| < M/2$ by (33) and (34). Combining (16), (18) and (19) yields

$$\|T_1^{k+1} y - x_T\| = \|T_1^{n_0} x - x_T\| \tag{35}$$

$$\leq \|T_1^{n_0} x - T^{n_0} x\| + \|T^{n_0} x - x_T\| < \varepsilon/8 + \varepsilon/8 < \varepsilon. \tag{36}$$

This completes the proof. ■

Our first main result comes as follows.

Theorem 2.13 (generic property of super-regular mappings in complete subspaces) *Let $(\mathcal{F}, d)$ be a complete metric space, $\mathcal{F} \subset \mathcal{N}(X)$ and $d \geq \rho$. Assume that the set of contraction mappings $\mathcal{C}$ is dense in $\mathcal{F}$. Then there exists a set $G \subset \mathcal{F}$ which is a countable intersection of open everywhere dense set in $\mathcal{F}$ such that each $T \in G$ is super-regular. In particular,* $\operatorname{Fix}(T) = (\operatorname{Id} - T)^{-1}(0) \neq \varnothing$ *is a singleton.*

Proof. By Proposition 2.12 and Proposition 2.11(i), for each $T \in \mathcal{C}$, in $(\mathcal{F}, d)$ there exists an open neighborhood $U(T, i)$ of T and an integer $n(T, i) \geq 2$ such that whenever $T_1 \in U(T, i)$, $n \geq n(T, i)$ and $x \in \mathbf{B}_i(0)$

$$\|T_1^n x - x_T\| < \frac{1}{i}. \tag{37}$$

Define $G := \bigcap_{q=1}^{\infty} O_q$ where

$$O_q := \bigcup \{U(T, i) : \ T \in \mathcal{C}, i = q, q+1, \ldots\}$$

which is dense and open in $\mathcal{F}$, since $\mathcal{C} \subset O_q$ and each $U(T, i)$ is open.

Let $T \in G$. Then there exists a sequence $(T_q)_{q=1}^{\infty}$ and a sequence $(i_q)_{q=1}^{\infty}$ with $i_q \geq q$ such that $T \in U(T_q, i_q)$ for $q = 1, 2, \ldots$. Then for each q, by (37), when $n \geq n(T_q, i_q)$ and $x \in \mathbf{B}_{i_q}(0)$ we have

$$\|T^n x - x_{T_q}\| < \frac{1}{i_q}. \tag{38}$$

It follows that when $n \geq \max\{n(T_q, i_q), n(T_p, i_p)\}$ and $\|x\| \leq \min\{i_p, i_q\}$,

$$\|x_{T_q} - x_{T_p}\| \leq \|x_{T_q} - T^n x\| + \|T^n x - x_{T_p}\| < \frac{1}{i_q} + \frac{1}{i_p},$$

thus $(x_{T_q})_{q=1}^{\infty}$ is a Cauchy sequence with a limit $x_T \in X$. Let $s > 0$ and $\varepsilon > 0$. Choose i_q and q sufficiently large such that $\mathbf{B}_s(0) \subset \mathbf{B}_{i_q}(0)$ and

$$\frac{1}{i_q} + \|x_{T_q} - x_T\| < \varepsilon.$$

In view of (38), for every $x \in \mathbf{B}_s(0)$ and $n \geq n(T_q, i_q)$ we have

$$\|T^n x - x_T\| \leq \|T^n x - x_{T_q}\| + \|x_{T_q} - x_T\| < \frac{1}{i_q} + \|x_{T_q} - x_T\| < \varepsilon.$$

Hence T is super-regular. The remaining result follows from Propostion 2.10. ■

Different choices of $\mathcal{F}$ lead to:

Theorem 2.14 (Reich & Zaslavaski [21]) *There exists a set $G \subset \mathcal{N}(X)$ which is a countable intersection of open everywhere dense sets in $\mathcal{N}(X)$ such that each $T \in G$ is super-regular.*

Proof. By Lemma 2.6(i), the set of contractions $\mathcal{C} \subset \mathcal{N}(X)$ is dense in $\mathcal{N}(X)$. Apply Theorem 2.13 to the complete metric space $(\mathcal{N}(X), \rho)$. ■

Theorem 2.15 (super-regularity of resolvents) *In $(\mathcal{J}(X), \hat{\rho})$, the set*

$$\{T \in \mathcal{J}(X) : \ T \textit{ is super-regular}\}$$

is residual.

Proof. By Lemma 2.6(ii), the set of contractions $\mathcal{C} \subset \mathcal{J}(X)$ is dense in $\mathcal{J}(X)$. Since

$$\hat{\rho}(T_1, T_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{2\|T_1 - T_2\|_n}{1 + 2\|T_1 - T_2\|_n} \geq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|T_1 - T_2\|_n}{1 + \|T_1 - T_2\|_n} = \rho(T_1, T_2) \quad \forall\, T_1, T_2 \in \mathcal{J}(X)$$

by (2) and (6), we have $\hat{\rho} \geq \rho$. It remains to apply Theorem 2.13 to the complete metric space $(\mathcal{J}(X), \hat{\rho})$. ■

Finding zeros of maximally monotone operators are important in optimization; see, e.g., [5, 12, 28, 17, 25]. However, we have

Theorem 2.16 (unique zero of monotone operators) *In $(\mathcal{M}(X), \tilde{\rho})$ there is a set $G \subset \mathcal{M}(X)$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}(X)$ such that each $A \in G$ has R_A super-regular. In particular, $A^{-1}(0) \neq \varnothing$ is a singleton.*

Proof. By Proposition 2.3, $(\mathcal{M}(X), \tilde{\rho})$ is isometric to $(\mathcal{N}(X), \rho)$. Apply Theorem 2.14 to $(\mathcal{N}(X), \rho)$ to obtain $\tilde{G}$ such that each $T \in \tilde{G}$ is super-regular and $\tilde{G}$ is a countable intersection of open everywhere dense sets in $\mathcal{N}(X)$. This $\tilde{G}$ corresponds to G in $(\mathcal{M}(X), \tilde{\rho})$

such that each $A \in G$ has R_A being super-regular and G is an intersection of open everywhere dense set in $\mathcal{M}(X)$. Note that $\mathrm{Fix}(R_A) = A^{-1}(0)$ by Proposition 1.3(iv). Since $\mathrm{Fix}(R_A)$ is a singleton when $A \in G$ by Proposition 2.10, the result holds. ■

In this connection, see also [8, Corollary 1], where De Blasi and Myjak showed a similar generic property for continuous and bounded monotone operators on a bounded set.

Corollary 2.17 *In $(\mathcal{M}(X), \tilde{\rho})$ there exists a set $G \subset \mathcal{M}(X)$ which is a countable intersection of open everywhere dense set in $\mathcal{M}(X)$ such that each $A \in G$ has both R_A and J_A being super-regular. In particular, $A^{-1}(0) \neq \varnothing$ is a singleton.*

Proof. Observe that for $A, B \in \mathcal{M}(X)$,

$$\tilde{\rho}(A,B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|R_A - R_B\|_n}{1 + \|R_A - R_B\|_n} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{2\|J_A - J_B\|_n}{1 + 2\|J_A - J_B\|_n} = \hat{\rho}(J_A, J_B)$$

by (5) and (6). Thus, $(\mathcal{M}(X), \tilde{\rho})$ and $(\mathcal{J}(X), \hat{\rho})$ are isometric under the mapping $A \mapsto J_A$. Apply Theorems 2.15 to $(\mathcal{J}(X), \hat{\rho})$ to obtain $\tilde{G_1} \subset \mathcal{J}(X)$ such that each $T \in \tilde{G_1}$ is super-regular. This $\tilde{G_1}$ corresponds to $G_1 \subset \mathcal{M}(X)$ such that each $A \in G_1$ has J_A being super-regular and G_1 is a countable intersection of open everywhere dense set in $\mathcal{M}(X)$. Apply Theorem 2.16 to obtain $G_2 \subset \mathcal{M}(X)$ such that each $A \in G_2$ has R_A being super-regular and G_2 is a countable intersection of open everywhere dense set in $\mathcal{M}(X)$. It suffices to let $G = G_1 \cap G_2$. ■

We finish this section with two examples.

Example 2.18 *For a maximal monotone operator $A \in \mathcal{M}(X)$, with regard to super-regularity a variety situations can happen to R_A and J_A.*

(1) Let $A : X \rightrightarrows X$ be given by $A = N_{\{0\}}$ the normal cone operator. Then $J_A = 0$ is super-regular, but $R_A = -\,\mathrm{Id}$ is not super-regular.

(2) Let $A : X \rightrightarrows X$ be given by $A = 0$ the zero operator. Then both $J_A = \mathrm{Id}$ and $R_A = \mathrm{Id}$ are not super-regular.

(3) Let $A : X \to X$ be given by $A = \mathrm{Id}$. Then $J_A = \mathrm{Id}/2$ and $R_A = 0$ are super-regular.

Example 2.19 *A super-regular mapping needs not be contractive.*

Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = |\sin x|$ for every $x \in \mathbb{R}$. Then T is not contractive but super-regular. T is not contractive because $\sup_{x \in R} |T'(x)| = 1$. To see that T is super-regular, we note that $0 \le Tx \le 1$ and for $n \ge 2$ the "iterative sequence" $(T^n)_{n=2}^{\infty}$ satisfies

$$0 \le T^n(x) = \sin(T^{n-1}(x)) \le T^{n-1}(x) \qquad \text{for every } x \in \mathbb{R}.$$

Being a decreasing monotone sequence bounded below, $(T^n(x))_{n=2}^{\infty}$ converges to 0, the unique fixed point of T. Since that the the decreasing function sequence $(T^n(x))_{n=1}^{\infty}$ con-

verges to 0 and that each T^n is continuous, T^n converges uniformly to 0 on every compact subset of R by the Dini's Theorem [26].

The results in the next section indicate that the set of contractive mappings and the set of strongly maximal monotone operators are too small.

3 Weakly contractive mapping, strong monotonicity and strong firmness

In this section we show that the set of contraction mappings in $(\mathcal{N}(X), \rho)$ (a subset of dense G_δ set in Theorem 2.14), the set of strongly monotone mappings in $(\mathcal{M}(X), \tilde{\rho})$ and the set of strongly firm nonexpansive mappings (a subset of dense G_δ set in Theorem 2.15) are first category, even they are dense in the corresponding metric spaces.

Definition 3.1 *A nonexpansive mapping $T \in \mathcal{N}(X)$ is weakly contractive if there exists $2 > l > 0$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - l(\|x - y\|^2 + \langle x - y, Tx - Ty \rangle) \quad \forall\, x, y \in X. \tag{39}$$

The set of weakly contractive mappings is strictly larger than the set of contractive mappings since $T = -\operatorname{Id}$ is weakly contractive but not contractive.

Definition 3.2 *A firmly nonexpansive mapping $T \in \mathcal{J}(X)$ is strongly firm nonexpansive if there exists $\varepsilon > 0$ such that*

$$(1 + \varepsilon)T \in \mathcal{J}(X),$$

in particular, T is $1/(1 + \varepsilon)$ contractive.

The following result states the relationship among R_A being weakly contractive, A being strongly monotone and J_A being strongly firmly nonexpansive.

Proposition 3.3 *Let $A \in \mathcal{M}(X)$. Then the following are equivalent:*

(i) *A is strongly monotone for some $\varepsilon > 0$;*

(ii) $\varepsilon \operatorname{Id} + (1 + \varepsilon)R_A$ *is nonexpansive;*

(iii) R_A *is $\frac{2\varepsilon}{1+\varepsilon}$ weakly contractive;*

(iv) $(1 + \varepsilon)J_A$ *is firmly nonexpansive.*

Proof. (i)$\Leftrightarrow$(ii): [6, Theorem 4.3]. (i)$\Leftrightarrow$(iv): [6, Theorem 2.1(xi)]. (ii)$\Leftrightarrow$(iii): (ii) means for $x, y \in X$,

$$\|\varepsilon x + (1+\varepsilon)R_A x - (\varepsilon y + (1+\varepsilon)R_A y)\|^2 \leq \|x-y\|^2,$$

that is,

$$\varepsilon^2\|x-y\|^2 + (1+\varepsilon)^2\|R_A x - R_A y\|^2 + 2\varepsilon(1+\varepsilon)\langle x-y, R_A x - R_A y\rangle \leq \|x-y\|^2.$$

Simple algebraic manipulation shows that this is equivalent to

$$\|R_A x - R_A y\|^2 \leq \|x-y\|^2 - \frac{2\varepsilon}{1+\varepsilon}(\|x-y\|^2 + \langle x-y, R_A x - R_A y\rangle).$$

■

Corollary 3.4 *Assume that $T \in \mathcal{N}(X)$ is a weakly contraction mapping for some $0 < l < 2$. Then $\operatorname{Fix}(T) \neq \varnothing$ and is a singleton.*

Proof. By Proposition 3.3, $T = R_A$ for a maximally monotone mapping and A is strongly maximal monotone. Since A is strongly monotone, we have $\operatorname{ran} A = X$ by Brezis-Haraux's range theorem [29, Corollary 31.6] so that $A^{-1}(0) \neq \varnothing$ and $A^{-1}(0)$ is a singleton. The proof is complete by using $\operatorname{Fix}(T) = A^{-1}(0)$. ■

Example 3.5 (1) *A weakly contractive mapping needs not be super-regular.* Let $A : X \rightrightarrows X$ be given by $A = N_{\{0\}}$ the normal cone operator. Then $R_A = -\operatorname{Id}$ weakly contractive but not super-regular.

(2) *A super-regular mapping needs not be weakly contractive.* From Example 2.19, the mapping

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto |\sin x|$$

is super-regular. Since $\frac{f+\operatorname{Id}}{2}$ is not contractive, f is not weakly contractive by Proposition 3.3.

Example 3.6 *A nonexpansive mapping can be neither weakly contractive nor super-regular.* On the Euclidean space $X = \mathbb{R}^2$, the $\pi/2$-degree rotator $T : X \to X$ given

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is neither weakly contractive nor super-regular. Indeed, T is nonexpansive since $\|Tx\| = \|x\|$ with $x \in X$; T is not weakly nonexpansive because (39) fails, as $\langle x-y, Tx-Ty\rangle = 0$ for $x, y \in X$; T is not super-regular because $||T^n x\| = \|x\|$ for every $x \in X, n \in \mathbb{N}$, and $T^n x \not\to 0$ unless $x = 0$.

The connection between weakly contractive mappings and contractive mappings comes next.

Proposition 3.7 *Let $T \in \mathcal{N}(X)$.*

(i) *If T is a contraction with modulus $0 \leq \beta < 1$, i.e., $\|Tx - Ty\| \leq \beta\|x-y\|$ for all $x, y \in X$, then both T and $-T$ are $(1-\beta)$ weakly contractive.*

(ii) *If both T and $-T$ are $(1-\beta)$ weakly contractive, then T is a contraction with modulus $\sqrt{\beta}$.*

Proof. (i): Assume that T is β contractive. For $x, y \in X$, by the Cauchy-Schwartz inequality and T being β contractive, we have

$$\langle x-y, Tx-Ty\rangle \leq \|x-y\|\|Tx-Ty\| \leq \beta\|x-y\|^2.$$

It follows that

$$\|x-y\|^2 - (1-\beta)(\|x-y\|^2 + \langle x-y, Tx-Ty\rangle) \tag{40}$$
$$\geq \|x-y\|^2 - (1-\beta)(\|x-y\|^2 + \beta\|x-y\|^2) \tag{41}$$
$$= \|x-y\|^2 - (1-\beta^2)\|x-y\|^2 = \beta^2\|x-y\|^2 \tag{42}$$
$$\geq \|Tx-Ty\|^2. \tag{43}$$

Hence T is $(1-\beta)$ weakly contractive. Applying to $-T$, we obtain that $-T$ is $(1-\beta)$ weakly contractive.

(ii): Assume that both T and $-T$ are $(1-\beta)$ weakly contractive. Then

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 - (1-\beta)(\|x-y\|^2 + \langle x-y, Tx-Ty\rangle) \quad \forall\, x, y \in X.$$

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 - (1-\beta)(\|x-y\|^2 - \langle x-y, Tx-Ty\rangle) \quad \forall\, x, y \in X.$$

Adding these inequality gives

$$2\|Tx-Ty\|^2 \leq 2\|x-y\|^2 - 2(1-\beta)\|x-y\|^2 = 2\beta\|x-y\|^2$$

which gives $\|Tx-Ty\| \leq \sqrt{\beta}\|x-y\|$ for $x, y \in X$. Hence T is a $\sqrt{\beta}$ contraction. ∎

It is very interesting to compare Proposition 3.3 to Fact 2.7.

Our main result in this section is

Theorem 3.8 (first category of weakly contraction mappings) *In $(\mathcal{N}(X), \rho)$, the set of weak contraction mappings, i.e., $\mathcal{K} :=$*

$$\left\{ \begin{array}{c} T \in \mathcal{N}(X) : \exists\, l > 0 \text{ such that } \|Tx-Ty\|^2 \leq \|x-y\|^2 - l(\|x-y\|^2 + \langle x-y, Tx-Ty\rangle) \\ \forall\, x, y \in X. \end{array} \right\}$$

is of first category.

Proof. Let $(l_n)_{n=1}^{\infty}$ be a positive strictly decreasing sequence in $(0,2)$ with $\lim_{n\to\infty} l_n = 0$. Define

$$(44)\quad \mathcal{K}_n := \left\{ \begin{matrix} T \in \mathcal{N}(X) : \|Tx-Ty\|^2 \le \|x-y\|^2 - l_n(\|x-y\|^2 + \langle x-y, Tx-Ty\rangle) \\ \forall\, x,y \in X. \end{matrix} \right\}.$$

Then $\mathcal{K}_{n+1} \supset \mathcal{K}_n$ for $n \in \mathbb{N}$ and $\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$. Clearly $\mathcal{K}_n$ is closed in $\mathcal{N}(X)$. We show that $\operatorname{int}\mathcal{K}_n = \varnothing$, where $\operatorname{int}\mathcal{K}_n$ stands for the interior of $\mathcal{K}_n$. Let $T \in \mathcal{K}_n$ and $\varepsilon > 0$. We will construct $T_2 \in \mathcal{N}(X)$ such that $\rho(T, T_2) \le 2\varepsilon$ and $T_2 \notin \mathcal{K}_n$. To this end, first apply Lemma 2.6 to find a contraction map T_1 with modulus $0 < L < 1$ such that $\rho(T_1, T) \le \varepsilon$. As $T_1 : X \to X$ is a contraction, it has a fixed point $x_0 \in X$. Next we follow the idea from De Blasi and Myjak [9]. Put

$$0 < \delta := \frac{(1-L)\varepsilon}{4} < \frac{\varepsilon}{4}.$$

Define

$$\tilde{T}_1(x) := \begin{cases} x & \text{if } x \in \mathbf{B}_\delta(x_0) \\ T_1(x) & \text{if } x \notin \mathbf{B}_{\varepsilon/2}(x_0). \end{cases}$$

Then $\tilde{T}_1$ is nonexpansive on $\mathbf{B}_\delta(x_0) \bigcup (X \setminus \mathbf{B}_{\varepsilon/2}(x_0))$. To see this, consider three cases: (i) If $x, y \in \mathbf{B}_\delta(x_0)$, then

$$\|\tilde{T}_1(x) - \tilde{T}_1(y)\| = \|x-y\|;$$

(ii) If $x, y \notin \mathbf{B}_{\varepsilon/2}(x_0)$, then

$$\|\tilde{T}_1(x) - \tilde{T}_1(y)\| = \|T_1x - T_1y\| \le L\|x-y\|;$$

(iii) $x \in \mathbf{B}_\delta(x_0), y \notin \mathbf{B}_{\varepsilon/2}(x_0)$. Note that $T_1x_0 = x_0$, T_1 being contractive with modulus L, and

$$\begin{aligned}
(45)\quad && \|x-y\| &= \|x - x_0 + x_0 - y\| \ge \|x_0 - y\| - \|x - x_0\| \\
(46)\quad && &\ge \frac{\varepsilon}{2} - \|x - x_0\| \ge \frac{\varepsilon}{2} - \delta \\
(47)\quad && &= \frac{\varepsilon}{2} - \frac{(1-L)\varepsilon}{4} = \frac{(1+L)\varepsilon}{4}.
\end{aligned}$$

It follows that

$$\begin{aligned}
(48)\quad && \|\tilde{T}_1(x) - \tilde{T}_1(y)\| &= \|x - T_1y\| \\
(49)\quad && &= \|x - x_0 + T_1x_0 - T_1x + T_1x - T_1y\| \\
(50)\quad && &\le \|x - x_0\| + \|T_1x_0 - T_1x\| + \|T_1x - T_1y\| \\
(51)\quad && &\le \|x - x_0\| + L\|x - x_0\| + L\|x-y\| \\
(52)\quad && &= (1+L)\|x - x_0\| + L\|x-y\| \\
(53)\quad && &\le (1+L)\delta + L\|x-y\| \\
(54)\quad && &= (1+L)\frac{(1-L)\varepsilon}{4} + L\|x-y\|
\end{aligned}$$

$$= (1-L)\frac{(1+L)\varepsilon}{4} + L\|x-y\| \quad (\text{using } (45)-(47)) \tag{55}$$

$$\leq (1-L)\|x-y\| + L\|x-y\| = \|x-y\|. \tag{56}$$

According to the Kirszbraun-Valentine extension theorem, see, e.g., [22], there exists a nonexpansive mapping $T_2 : X \to X$ extending $\tilde{T}_1$ from $\operatorname{dom} \tilde{T}_1$ to X.

Claim 1: $\rho(T_2, T_1) \leq \varepsilon$.

To see this, observe that $T_2 x = \tilde{T}_1(x) = T_1(x)$ if $x \in X \setminus \mathbf{B}_{\varepsilon/2}(x_0)$. When $x \in \mathbf{B}_\delta(x_0)$ we have

$$\|T_2 x - T_1(x)\| = \|x - T_1 x\| = \|x - x_0 + T_1 x_0 - T_1 x\| \tag{57}$$

$$\leq \|x - x_0\| + \|T_1 x_0 - T_1 x\| \leq \|x - x_0\| + L\|x - x_0\| \tag{58}$$

$$= (1+L)\|x - x_0\| \leq (1+L)\delta = (1+L)\frac{(1-L)\varepsilon}{4} \leq \varepsilon; \tag{59}$$

When $x \in \mathbf{B}_{\varepsilon/2}(x_0) \setminus \mathbf{B}_\delta(x_0)$, pick

$$y := x_0 + \frac{\varepsilon}{2}\frac{y - x_0}{\|y - x_0\|}$$

so that $y \in \mathbf{B}_{\varepsilon/2}(x_0)$ and $T_2 y = T_1 y$. We have

$$\|T_2 x - T_1 x\| = \|T_2 x - T_1 x - (T_2 y - T_1 y)\| = \|(T_2 x - T_2 y) - (T_1 x - T_1 y)\| \tag{60}$$

$$\leq \|T_2 x - T_2 y\| + \|T_1 x - T_1 y\| \tag{61}$$

$$\leq \|x-y\| + L\|x-y\| = (1+L)\|x-y\| \leq (1+L)(\varepsilon/2 - \delta) \leq \varepsilon. \tag{62}$$

Then

$$\rho(T, T_2) \leq \rho(T, T_1) + \rho(T_1, T_2) \leq 2\varepsilon.$$

Claim 2: $T_2 \notin \mathcal{K}_n$. This is because $T_2 x = x$ for $x \in \mathbf{B}_\delta(x_0)$.

Since ε was arbitrary, $\operatorname{int} \mathcal{K}_n = \varnothing$. This completes the proof. ∎

Combing Theorem 3.8 and Proposition 3.7(i) immediately yields

Corollary 3.9 (first category of contraction mappings) *In $(\mathcal{N}(X), \rho)$, the set of contractive mappings, i.e., $\mathcal{K} =$*

$$\{T \in \mathcal{N}(X) : \exists\, 1 > l \geq 0 \text{ such that } \|Tx - Ty\| \leq l\|x-y\| \;\forall\, x, y \in X\}$$

is of first category.

While a similar result for nonexpansive mappings defined on a closed bounded convex set $C \subset X$ was obtained by De Blasi and Myjak in [9] and Reich [19], Corollary 3.9 concerns nonexpansive mappings on a unbounded set X.

There are many ways to generate strongly monotone mappings: $A + \varepsilon\,\mathrm{Id}$ (Tychonov regularization), $S_A^\varepsilon = \mathcal{R}(A, \mathrm{Id}, 1-\varepsilon, \varepsilon)$ (self-dual regularization), see, e.g., [30]. Corresponding results for maximal monotone operators and firmly nonexpansive mappings follow at once by combing Proposition 3.3 and Theorem 3.8.

Corollary 3.10 (first category of strongly monotone mappings) *In $(\mathcal{M}(X), \tilde{\rho})$, the set*

$$\{A \in \mathcal{M}(X) : \exists\, \varepsilon > 0 \text{ such that } A \text{ is } \varepsilon \text{ strongly monotone}\}$$

is of first category.

Corollary 3.11 (first category of strongly firm nonexpansive mappings) *In $(\mathcal{J}(X), \hat{\rho})$, the set*

$$\{T \in \mathcal{J}(X) : \exists\, \varepsilon > 0 \text{ such that } (1+\varepsilon)T \text{ is firmly nonexpansive}\}$$

is of first category.

Appendix

For $C \subset X$, $\overline{C}$ denotes its norm closure. The proofs to Theorems 2.13, 2.15 and 2.16 are harder, and rely on Reich and Zaslavaski's super-regularity mappings. If one only wants $0 \in \overline{\mathrm{ran}(\mathrm{Id} - T)}$, $0 \in \overline{\mathrm{ran}\, A}$ and asymptotic regularity of J_A (much weaker results), a much simpler argument works. This is the purpose of this appendix.

Theorem 3.12 (almost fixed point of nonexpansive mapping) *The set*

$$G := \{T : 0 \in \overline{\mathrm{ran}(\mathrm{Id} - T)}\} \tag{63}$$

is dense G_δ in $(\mathcal{N}(X), \rho)$. Thus, generically nonexpansive mappings almost have fixed points.

Proof. For every $n \in \mathbb{N}$ define

$$O_n := \left\{ T \in \mathcal{N}(X) : \text{ there exists } x \in X \text{ such that } \|x - Tx\| < \frac{1}{n} \right\}.$$

Claim 1. O_n is dense. Let $T \in \mathcal{N}(X)$ and $\varepsilon > 0$. Apply Lemma 2.6 to find a contraction T_2 such that $\rho(T, T_2) < \varepsilon$. Since T_2 is a contraction, it has a fixed point by the Banach Contraction Principle [16, Theorem 5.1.2], thus $T_2 \in O_n$. Therefore, O_n is dense in $\mathcal{N}(X)$.

Claim 2. O_n is open. Let $T \in O_n$. Then there exists $x \in X$ such that

$$\|x - Tx\| < \frac{1}{n}. \tag{64}$$

Assume that $K \in \mathbb{N}$ and $\|x\| < K$. Put

$$r = \frac{1}{2^K}\frac{1/n - \|x - Tx\|}{1 + 1/n - \|x - Tx\|}.$$

We show that $\mathbf{B}_r(T) := \{T_1 \in \mathcal{N}(X) : \rho(T_1, T) < r\} \subset O_n$. Let $T_1 \in \mathbf{B}_r(T)$. Since $\rho(T_1, T) < r$, we have

$$\frac{1}{2^K}\frac{\|T_1 - T\|_K}{1 + \|T_1 - T\|_K} \leq \rho(T_1, T) < \frac{1}{2^K}\frac{1/n - \|x - Tx\|}{1 + 1/n - \|x - Tx\|}$$

so that

$$\frac{\|T_1 - T\|_K}{1 + \|T_1 - T\|_K} < \frac{1/n - \|x - Tx\|}{1 + 1/n - \|x - Tx\|}.$$

It follows that

$$\|T_1 - T\|_K < \frac{1}{n} - \|x - Tx\|.$$

Then using $\|x\| \leq K$,

$$\|x - T_1x\| = \|x - Tx + Tx - T_1x\| \leq \|x - Tx\| + \|Tx - T_1x\| \tag{65}$$

$$\leq \|x - Tx\| + \|T - T_1\|_K < \|x - Tx\| + \frac{1}{n} - \|x - Tx\| = \frac{1}{n}. \tag{66}$$

Therefore $T_1 \in O_n$. Since $T_1 \in \mathbf{B}_r(T)$ was arbitrary, $\mathbf{B}_r(T) \subset O_n$.

As $(\mathcal{N}(X), \rho)$ is a complete metric space, $\bigcap_{n=1}^{\infty} O_n$ is a dense G_δ set in $\mathcal{N}(X)$ by Fact 1.6. If $T \in \bigcap_{n=1}^{\infty} O_n$, then for every $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$\|x_n - Tx_n\| < \frac{1}{n}$$

thus $0 \in \overline{\operatorname{ran}(\operatorname{Id} - T)}$. Hence $\bigcap_{n=1}^{\infty} O_n \subset G$. On the other hand, if $T \in G$, then $0 \in \overline{\operatorname{ran}(\operatorname{Id} - T)}$. It follows that for every n there exists $x_n \in X$ such that $\|x_n - Tx_n\| < 1/n$ so $T \in O_n$. As this holds for every n and $T \in G$, we have $G \subset \bigcap_{n=1}^{\infty} O_n$. Altogether, $G = \bigcap_{n=1}^{\infty} O_n$ which is a dense G_δ set in $\mathcal{N}(X)$. This completes the proof. ■

Theorem 3.13 (almost zeros of maximal monotone operator) *In $(\mathcal{M}(X), \tilde{\rho})$, the set*

$$\{A \in \mathcal{M}(X) :\ 0 \in \overline{\operatorname{ran} A}\} \tag{67}$$

is a dense G_δ set. Hence, generically maximally monotone operators almost have zeros.

Proof. By Theorem 3.12 and Proposition 1.3(i), the set

$$\{A \in \mathcal{M}(X) : 0 \in \overline{\operatorname{ran}(\operatorname{Id} - R_A)}\}$$

is dense G_δ in $(\mathcal{M}(X), \tilde{\rho})$. Observe that

$$\operatorname{ran}(\operatorname{Id} - R_A) = \operatorname{ran}(2\operatorname{Id} - 2J_A) = 2\operatorname{ran}(\operatorname{Id} - J_A) = 2\operatorname{ran} J_{A^{-1}} = 2\operatorname{dom} A^{-1} = 2\operatorname{ran} A.$$

Hence (67) holds. ■

Recall that $T : X \to X$ is *asymptotically regular* at x if $\lim_{n\to\infty}(T^{n+1}x - T^n x) = 0$, cf. [10, 5]. Asymptotic regularity is one of critical properties in many iterative algorithms, [5, page 79], [2],

Theorem 3.14 (asymptotic regularity of resolvent) *In $(\mathcal{J}(X), \hat{\rho})$, the set*

$$\{T \in \mathcal{J}(X) : \ \|T^{n+1}x - T^n x\| \to 0 \ \forall x \in X\} \tag{68}$$

is a dense G_δ set. Consequently, generically resolvents are asymptotically regular.

Proof. Each $T \in \mathcal{J}(X)$ is firmly nonexpansive, so strongly nonexpansive. By [10, Corollary 1.5] Bruck and Reich,

$$\lim_{n\to\infty}(T^n x - T^{n+1}x) = v \tag{69}$$

where v is the smallest norm element of $\overline{\operatorname{ran}(\operatorname{Id} - T)}$. It follows for Theorem 3.12 and (7) that the set

$$\{T \in \mathcal{J}(X) : \ 0 \in \overline{\operatorname{ran}(\operatorname{Id} - (2T - \operatorname{Id}))}\}$$

i.e.,

$$\{T \in \mathcal{J}(X) : \ 0 \in \overline{\operatorname{ran}(\operatorname{Id} - T)}\}$$

is a dense G_δ set in $\mathcal{J}(X)$. It suffices to apply (69). ■

Acknowledgments

I would like to thank Dr. Heinz Bauschke for his constructive suggestions and comments on the paper. Xianfu Wang was partially supported by the Natural Sciences and Engineering Research Council of Canada.

References

[1] H. Attouch, *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman Advanced Publishing Program, Boston, MA, 1984.

[2] J.B. Baillon, R.E. Bruck and S. Reich, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston J. Math.* 4 (1978), 1–9.

[3] S. Bartz, H.H. Bauschke, J.M. Borwein, S. Reich, and X. Wang, Fitzpatrick functions, cyclic monotonicity and Rockafellar's antiderivative, *Nonlinear Anal.* 66 (2007), 1198–1223.

[4] H.H. Bauschke, R.I. Bot, W.L. Hare, W.M. Moursi, Attouch-Théra duality revisited: Paramonotonicity and operator splitting, *J. Approx. Theory* 164 (2012), 1065–1084.

[5] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, 2011.

[6] H.H. Bauschke, S.M. Moffat and X. Wang, Firmly nonexpansive mappings and maximally monotone operators: correspondence and duality, *Set-Valued Var. Anal.* 20 (2012), 131–153.

[7] F.S. De Blasi and J. Myjak, Sur la porosité de l'ensemble des contractions sans point fixe, (French) [On the porosity of the set of contractions without fixed points], *C. R. Acad. Sci. Paris Sér. I Math.* 308 (1989), 51–54.

[8] F.S. De Blasi and J. Myjak, Generic properties of contraction semigroups and fixed points of nonexpansive operators, *Proc. Amer. Math. Soc.* 77 (1979), 341–347.

[9] F.S. De Blasi and J. Myjak, Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach, *C. R. Acad. Sci. Paris* 283 (1976), 185–187.

[10] R.E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston J. Math.* 3 (1977), 459–470.

[11] P.L. Combettes and T. Pennanen, Generalized Mann iterates for constructing fixed points in Hilbert spaces, *J. Math. Anal. Appl.* 275 (2002), no. 2, 521–536.

[12] P.L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization* 53 (2004), 475–504.

[13] J. Eckstein and D.P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, *Math. Programming* 55 (1992), 293–318.

[14] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, 1984.

[15] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.

[16] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New-York, 1978.

[17] P.L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.* 16 (1979), no. 6, 964–979.

[18] G.J. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J.*, 29 (1962), 341–346.

[19] S. Reich, Genericity and porosity in nonlinear analysis and optimization, *ESI Preprint 1756, 2005*, Proceedings of CMS'05 (Computer Methods and Systems), Krakow, 2005, 9–15.

[20] S. Reich and A. J. Zaslavski, Generic aspects of metric fixed point theory, *Handbook of Metric Fixed Point Theory*, Kluwer, Dordrecht, 2001, 557–575.

[21] S. Reich and A. J. Zaslavski, Convergence of Krasnoselskii-Mann iterations of nonexpansive operators, *Math. Comput. Modelling* 32 (2000), 1423–1431.

[22] S. Reich and S. Simons, Fenchel duality, Fitzpatrick functions and the Kirszbraun-Valentine extension theorem, *Proc. Amer. Math. Soc.* 133 (2005), 2657–2660.

[23] S. Reich and A.J. Zaslavski, Almost all nonexpansive mappings are contractive, *C. R. Math. Acad. Sci. Soc. R. Can.* 22 (2000), 118–124.

[24] S. Reich and A.J. Zaslavski, The set of noncontractive mappings is σ-porous in the space of all nonexpansive mappings, *C. R. Acad. Sci. Paris Sr. I Math.* 333 (2001), 539–544.

[25] R.T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.*, 14 (1976), 877–898.

[26] H.L. Royden, *Real Analysis*, Prentice Hall, 3rd edition, 1988.

[27] R.T. Rockafellar and R. J-B Wets, *Variational Analysis*, Springer, corrected 3rd printing, 2009.

[28] S. Sabach, Products of finitely many resolvents of maximal monotone mappings in reflexive Banach spaces, *SIAM J. Optim.* 21 (2011), 1289–1308.

[29] S. Simons, *From Hahn-Banach to Monotonicity*, Springer, 2008.

[30] X. Wang, Self-dual regularization of monotone operators via the resolvent average, *SIAM J. Optim.* 21 (2011), 438–462.

[31] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B: Nonlinear Monotone Operators*, Springer-Verlag, 1990.